Experiments on Joule Heating and the Dissipation of Energy in the Earth's Core

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SUMMARY

We present measurements of Joule heat production in a fluid gallium vortex permeated by a uniform transverse magnetic field. We find that the Joule heat production increases as the square of the imposed field intensity for weak and moderately strong magnetic fields and magnetic Reynolds numbers up to about one. For stronger magnetic fields, Lorentz forces destroy the two-dimensional structure of the vortex and the Joule heat production becomes nearly independent of the intensity of the imposed magnetic field. We derive scaling laws relating fluid velocity in the vortex, imposed magnetic field intensity and Joule heat production, for both low and high magnetic Reynolds number regimes. Application of these laws to magnetic induction in the Earth's fluid core indicates that Joule heat production by this mechanism is large enough to limit the intensity of magnetic fields within the core.

RESUMÉ

Dans cet article, nous présentons des mesures de puissance de dissipation ohmique (effet Joule) effectuées dans un vortex de gallium soumis à un champ magnétique uniforme transversal. Nous trouvons que la production de chaleur est proportionnelle au carré de l'intensité du champ magnétique appliqué, pour des champs faibles et modérés, et pour un nombre de Reynolds magnétique inférieur à 1. Pour des champs plus forts, les forces de Lorentz détruisent la bidimensionalité du vortex, et la production de chaleur par effet Joule devient pratiquement indépendante de l'intensité du champ magnétique appliqué. Nous établissons des lois d'échelles qui relient la vitesse du fluide dans le vortex, l'intensité du champ magnétique appliqué, et la puissance ohmique dissipée, pour les regimes à bas et haut nombre de Reynolds magnétique. L'application de ces lois à l'induction magnétique dans le noyau terrestre indique que la production de chaleur par ce mécanisme est assez grande pour limiter l'intensité du champ magnétique dans le noyau.

Key words:

Joule Heating, Earth's core, Earth's magnetic field, Gallium, M.H.D. experiments.

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1 Introduction

The geomagnetic field is continually regenerated by motions of the electrically conducting iron-rich outer core. Although the concept of a self-sustaining dynamo is widely accepted for the core (Loper and Roberts, 1983; Braginski, 1990; Cardin and Olson, 1992), many of the critical parameters are unresolved. For example the intensity of the magnetic field on the core-mantle boundary is known to be about 0.5 mT (Voorhies, 1986; Bloxham, Gubbins and Jackson, 1989), but inside the electrically conducting core the field intensity is essentially unknown. Some dynamo theories predict intense toroidal fields in the core, of the order 1000 mT (Kumar and Roberts, 1975; Braginski, 1990), whereas other theories predict the core toroidal field is only about as strong as the dipole field (Pekeris, Accad and Shkoller, 1973; Busse, 1975). The Glatzmaier and Roberts (1995) numerical simulation of thermal convection in a rotating, electrically-conducting fluid sphere produced an external magnetic field with an amplitude similar to the Earth's dipolar field with an internal field of the order 10 mT. Zhang and Fearn (1993) have argued that diffusive instabilities limit the strength of the toroidal field to about 10 mT or less. It is clear from the spread of values among these estimates that the toroidal field is very poorly constrained.

Another important but poorly understood process is the mechanism by which the dynamo dissipates energy. It is generally agreed that the kinetic energy in the core is transformed into heat primarily by ohmic, rather than viscous dissipation (Verhoogen, 1980). The rate of Ohmic dissipation is the Joule heat production, given by

$$\Phi = \int_{core} \frac{J \cdot J}{\sigma} dV \tag{1}$$

where J is the current density and σ is the electrical conductivity. Since

$$\mu_0 J = \nabla \times B \tag{2}$$

where B is the magnetic induction, the process of energy dissipation is closely related to the intensity and the structure of the magnetic field in the core.

Theoretical considerations indicate the dynamo is dissipation-limited, which implies there is a close relationship between the energy available to drive the dynamo and the energy dissipated by the dynamo. For example, in convection-driven dynamos the dissipation is proportional to the convective heat transport Q via

$$\Phi = \eta Q \tag{3}$$

where $\eta \simeq 0.1$ is the thermodynamic efficiency factor. Since the core energy budget limits the convective heat transport to roughly $Q \leq 5$ TW (Verhoogen, 1980; Lister and Buffet, 1995), the Joule heat production is limited to about $\Phi \leq 0.5$ TW.

The amount of Joule heating associated with the present-day dipole moment of the geomagnetic field is only about 10^{-2} TW, well below the thermodynamic limit. However, it is likely that Joule heating in the core is concentrated in smaller-scale current systems that do not contribute directly to the dipole moment (Gubbins, Masters and Jacobs, 1979). Evidence for dissipation in smaller-scale electrical currents comes from the spectrum of the geomagnetic field on the core-mantle boundary (CMB). Excluding the dipole term, the power spectrum of the radial field on the cmb is nearly constant out to spherical harmonic degree $l = 13$, the limit of resolution of the core field (Hulot and Le Mouël, 1994). Since the magnetic energy spectrum is nearly constant, the energy dissipation spectrum may actually *increase with spherical harmonic degree* over this range, implying that Joule heating is larger at higher degrees, that is, at short length scales.

The experiments we present here illustrate this effect, demonstrating that the Joule heating occurs on the internal length scales of the flow, rather than the length scale of the core as a whole. We measure the dissipation in a fluid vortex permeated by a transverse magnetic field. This serves as a simplified model for dissipation in the core: the vortex represents a single geostrophic convection column with its axis parallel with the axis of rotation (Busse, 1970; Cardin and Olson, 1994; Brito, Cardin, Nataf and Marolleau, 1995) and the transverse magnetic field represents the toroidal magnetic field in the core.

A previous study (Brito et al., 1995), examined the dynamics of a mechanically-driven geostrophic vortex of gallium in a transverse magnetic field. In that study, the effect of the Lorentz force on the circulation and radius of the vortex was determined, and the pattern of the induced magnetic field was delineated.

Here, we use the same apparatus to measure Joule heating that results from the interaction of the vortex with the imposed magnetic field. The experimental results, up to magnetic Reynolds number of 0.3, are compared with two simplified theoretical models of Joule heating, valid at low and high magnetic Reynolds number, respectively, and then extrapolated to the parameter regime of the Earth's core.

The plan of this paper is as follows. We describe the experiment design in section 2, and the Joule heating results are given in section 3. A simple scaling law for these results is derived in section 4, together with an analytical extension of this law for very large magnetic Reynolds number. In section 5, we extrapolate our results to estimate the dissipation by convection in the Earth's core. Conclusions and perspectives are presented in section 6.

2 Experimental approach

2.1 Experimental set-up

The experimental set-up is essentially the same as in Brito et al. (1995), hereafter referred to as paper I. Figure 1 illustrates the main features. A polycarbonate cylinder [8 cm (inner diameter) $\times$ 22 cm (inner height)] is filled with liquid gallium (one liter), and is mounted vertically between the poles of an electromagnet. The magnet produces a uniform horizontal magnetic field between the two 16 cm diameter poles with a maximum intensity of 80 mT. The magnet axis passes 13 cm above the base of the cylinder.

A fluid vortex with a vertical axis is created in the gallium by the steady rotation of a 4 cm diameter crenellated disk, located 3.8 cm above the base of the cylinder. The disk is connected to a speed and torque controlled motor, via a shaft and drive belt system. The motor is shielded from the fluid in order to avoid interference with the applied or induced magnetic fields.

Temperature in the fluid is recorded by a thermistor located near the top of the cylinder as shown in Figure 1. Two other thermistors located in the polycarbonate base plate are used to monitor the effects of friction at the rotary joint where the motor shaft enters the cylinder. In addition to measuring temperature, we also measure the horizontal component of the induced field, in the direction perpendicular to the applied field using a gaussmeter positioned just outside of the cylinder polycarbonate, at a height of 13 cm above its base.

There are several differences between the set-up of this experiment and the experiment reported in paper I. Among these are: i) the cylinder and applied magnetic field are at rest in the laboratory frame, whereas they were rotating in paper I; ii) in this experiment we use disk velocities up to 1500 rev min^{-1}, where velocities less than 600 rev min^{-1} were used in paper I; and iii) the top cover that received an array of Venturi tubes in paper I is replaced by a plain cover with a Pt-100 thermistor (3 mK precision) attached to it.

2.2 Measurement of Heat Production in Liquid Gallium

The principle of our experiments is very simple: we measure the heat production from dissipative processes in the fluid by monitoring the increase in temperature of the liquid gallium as a

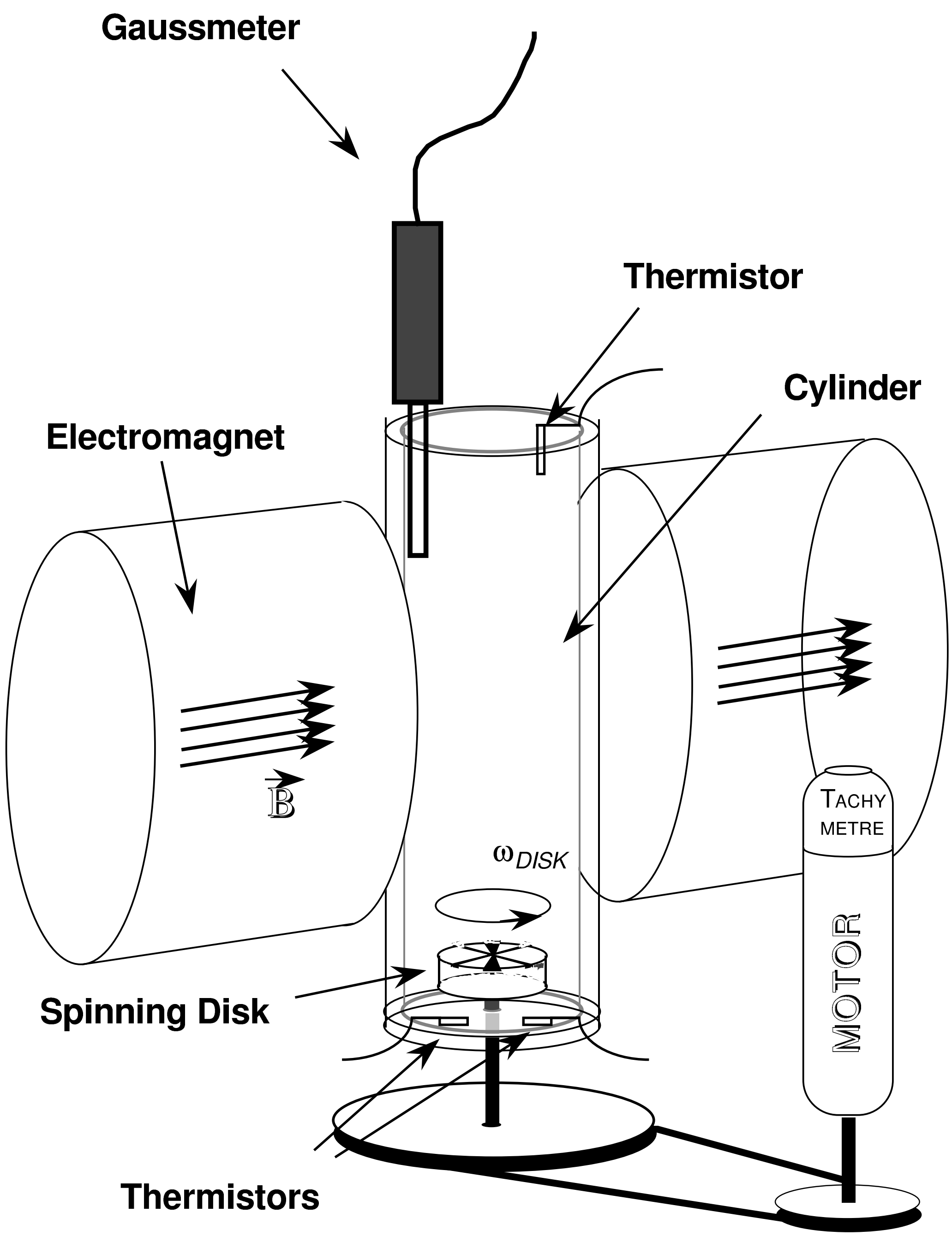

Figure 1: Sketch of the experimental set-up.

function of time. If the cylinder was perfectly insulated and if Joule heating the only source of dissipation, we would have:

$$P_J = mC\frac{d\bar{T}}{dt} \tag{4}$$

where P_J is Joule power, m is the mass of fluid, C is the specific heat and $\bar{T}$ is the average temperature of gallium. In reality, the experiment involves additional sources of heat and the polycarbonate container is not perfectly insulating. The most important complicating effects consist of heat loss P_{loss} through the insulated walls of the cylinder, heat production by viscous dissipation in the liquid P_μ, and frictional heating by the shaft of the spinning disk P_{shaft}. Including these effects leads to

$$mC\frac{d\bar{T}}{dt} = P_J + P_\mu + P_{shaft} - P_{loss} \tag{5}$$

We estimated the relative importance of each of these effects using a series of preliminary tests of the device. The dominant heat source turns out to be P_{shaft}, which is typically a factor of ten larger than the dissipation within the fluid.

In order to eliminate the contribution of P_{shaft} and P_{loss} in (5), we use *differential* measurements of temperature increase, as follows. Prior to and during each experiment, the cylinder is maintained at a temperature of about 40 °C by a circulating hot air system. To begin each experiment, the disk is spun-up from rest to a constant angular velocity ω_{disk} and maintained at this velocity until the temperature near the shaft (measured by two auxiliary thermistors shown in Figure 1) is 2 °C higher than the temperature in the gallium. We then switch on the electromagnet, applying a steady magnetic field with intensity $B_{imposed}$ and then record the fluid temperature at 10 seconds intervals for about 3 minutes. We then switch off the electromagnet and continue recording temperature for another 3 minutes, again at 10 seconds intervals.

Since the liquid is well stirred by the swirling vortex, the temperature measured at the top approximates very closely the average temperature of the gallium. Using the *difference* in the rate of temperature increase with and without magnetic field eliminates the contribution from P_{shaft}. It also largely eliminates the contribution to (5) from imperfect insulation, since P_{loss} is proportional to the temperature difference between the liquid and the outside temperature, which is almost constant during a complete run. Finally, viscous dissipation is found to be negligible compared with Ohmic dissipation, as anticipated (Tritton, 1988). We conducted a series of tests in which we systematically increased the field of the electromagnet in steps, and

Density	ρ	kg/m^3	6.09×10^3
Kinematic viscosity	ν	m^2/s	3.1×10^{-7}
Electrical conductivity	σ	$(m\Omega)^{-1}$	3.68×10^6
Melting point	T_m	°C	29
Boiling point	T_b	°C	2227
Coefficient of thermal expansion	α	K^{-1}	$1.0\ \times10^{-4}$
Surface tension	γ	N/m	0.735
Specific heat	C	(J/kg)/K	410

Table 1: Physical properties of liquid gallium.

extrapolated the heating rate obtained by the differential temperature measurements to zero field strength. The extrapolated zero-field heating rate is exceedingly small, typically less than 1% of the heating rates shown in Figure 3. This indicates that viscous dissipation in the fluid is negligible compared to Ohmic dissipation, and that the difference in rate of temperature rise with versus without the field of the electromagnet measures the Joule heating in the fluid.

Figure 2 shows a typical record of temperature in the fluid obtained using this technique. The temperature time series is piece-wise linear, with a clear decrease in slope when the magnetic field is switched off. According to the arguments just given, the difference in slope yields the Joule heating P_J as a function of ω_{disk} and $B_{imposed}$ as follows:

$$P_J = mC\left[\left(\frac{dT}{dt}\right)^{B=B_{imposed}}_{\omega=\omega_{disk}} - \left(\frac{dT}{dt}\right)^{B=0}_{\omega=\omega_{disk}}\right] \qquad (6)$$

In evaluating the r.h.s. of this expression, we determine the slopes of the two segments of $\frac{dT}{dt}$ using the least-square fits of linear equations to the data points, and then compute the difference in the two slopes.

The physical properties of gallium required for the analysis are listed in Table 1 (Pascal, 1961).

3 Experimental results

3.1 Joule heating

Joule heating was measured with the techniques described above for disk velocities ω_{disk} of 600, 1000, 1250 and 1500 rev min^{-1}, and applied magnetic fields from 0 to 80 mT. Figure 3 displays the results. The error bars are the 2σ standard deviations deduced from the linear fits to the temperature time series.

The data points for ω_{disk} = 1500 rev min^{-1} illustrate the behavior we find generally. These data exhibit two regimes. In the first regime, for magnetic field up to about 40 mT, Joule

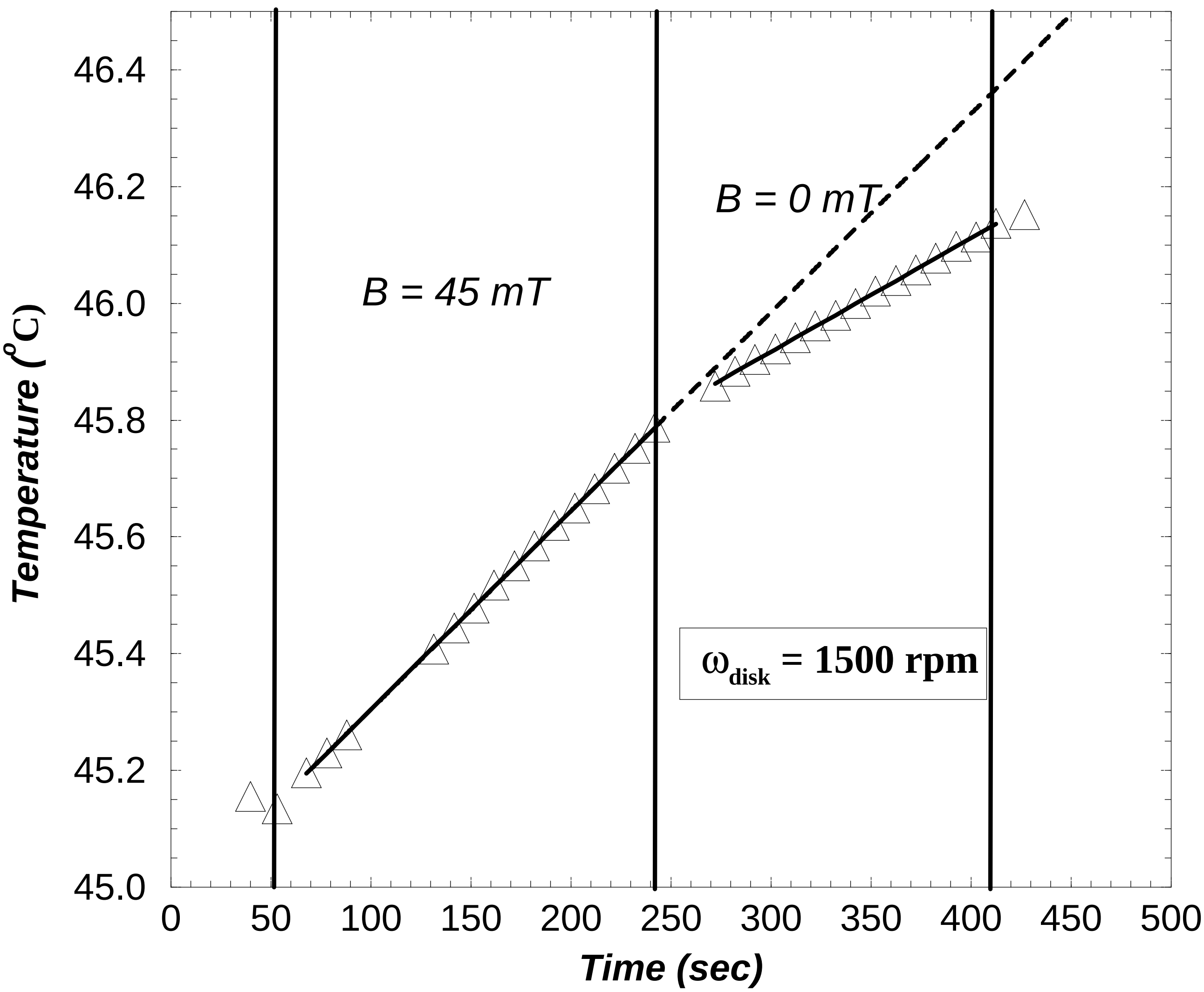

Figure 2: Temperature increase measured for $\omega_{disk} = 1500$ rev min^{-1}. From 50 s. to 250 s. a magnetic field of 45 mT is applied, and from 250 s. to 410 s. the electromagnet is switched off. Each triangle (every 10 s.) is one measurement of the temperature at the top of the vortex. The continuous lines are the slopes of the least squares straight fits for $\left(\frac{dT}{dt}\right)^{B=45}_{\omega=1500}$ and $\left(\frac{dT}{dt}\right)^{B=0}_{\omega=1500}$ of equation (6). We can notice the difference in slopes between the dashed line and the solid line during the second part of the experiment (after 250 s.): the increase of temperature in the tank of gallium is larger when the magnetic field is applied because of the heat dissipated by the electrical currents (Joule heating).

heating increases nearly quadratically with the field intensity. Beyond this value is a second regime where Joule heating levels off and perhaps even decreases with increasing field strength. For lower values of ω_{disk}, a similar behavior is seen, except that the transition between the two regimes occurs at a corresponding lower value of magnetic field.

The existence of two regimes in Joule heating is consistent with the experimental findings in paper I. In I, measurements of pressure profiles at the top of the vortex, electrical potentials, and induced magnetic field, identified two distinct flow regimes. In those experiments, the controlling parameter was found to be the Elsasser number, the ratio of Lorentz over Coriolis forces. It was observed that for low Elsasser number, less than 0.2 approximately (corresponding to low imposed magnetic fields), the vortex is slowed down by the magnetic field, but remains essentially two-dimensional and extends throughout the height of the cylinder. The effective diameter of the vortex increases with the imposed field in this regime. At higher values of the imposed field (or, alternatively smaller values of Ω_{Table}), corresponding to larger Elsasser numbers, the vortex is nearly arrested by the magnetic field, and the basic two-dimensionality of the flow is destroyed. For increasing field strengths, the motion in the fluid is increasingly confined to the vicinity of the spinning disk in this regime. In the next section we will present a quantitative analysis of our results on Joule heating in terms of the vortex velocity, and demonstrate that the transition between these two flow regimes explains the Joule heating data.

3.2 Torque measurements

In addition to measuring the heat dissipated within the fluid, we monitored the torque applied by the motor that drives the disk. Torque variations were recorded as a function of time in all the experiments. Using the same differential measurement technique as described above for heating, we have to obtain the torque Γ_L applied by the fluid on the disk when Lorentz forces are present. A simple energy balance indicates that Ohmic dissipation within the fluid is equal to the work done on the fluid by Lorentz forces and is thus related to the torque driving the disk by:

$$P_J = \Delta\Gamma_L \, \omega_{disk} \tag{7}$$

where $\Delta\Gamma_L$ is the torque difference between the portions of the experiment with and without the field of the electromagnet imposed. In principle, this provides an alternative measurement of Joule heating. We find that the dissipation measured this way depends on the field intensity

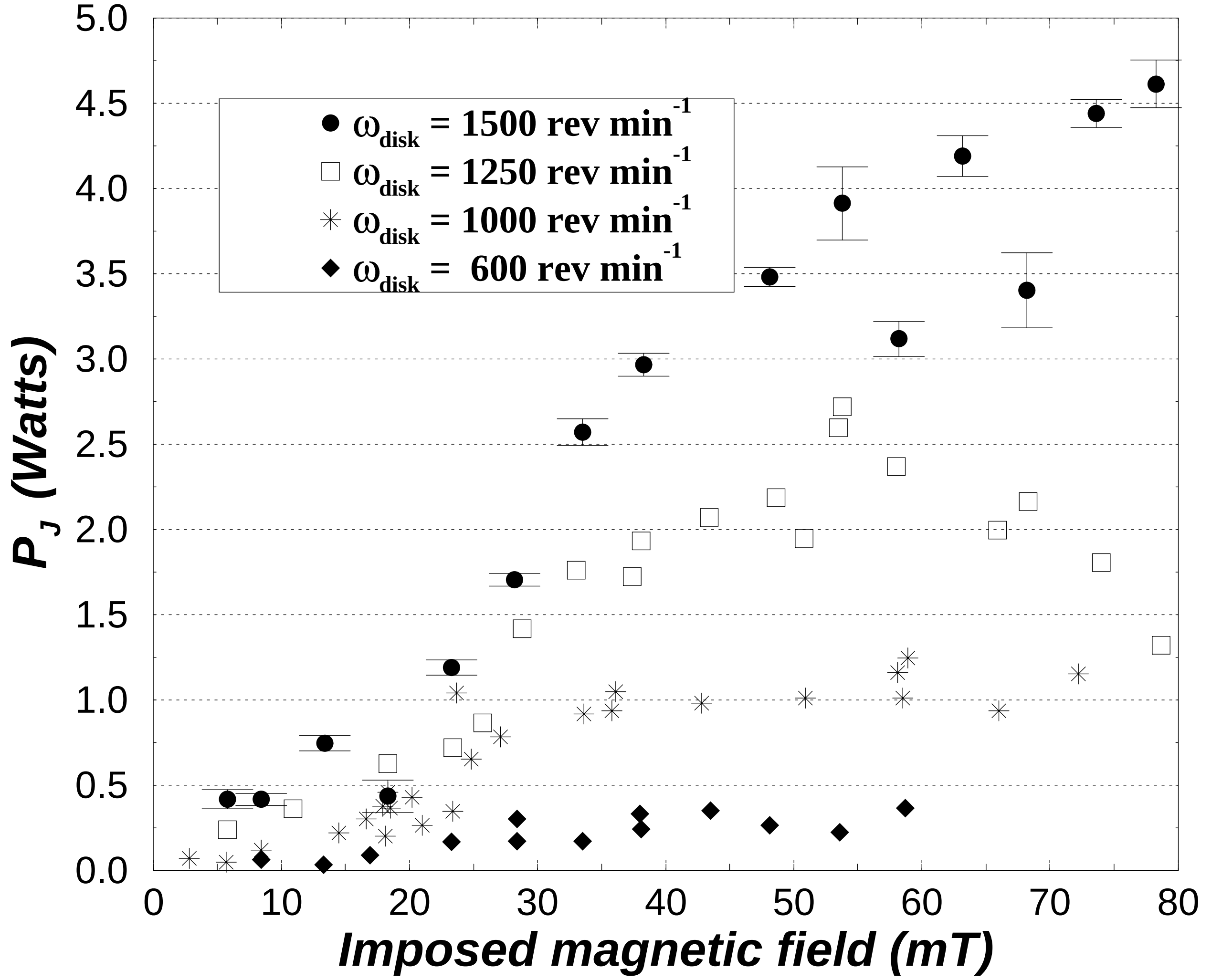

Figure 3: Joule Heating measured for ω_{disk} =600, 1000, 1250 and 1500 rev min^{-1}. We clearly observe two regimes for the four different velocities. However, the second regime is reached for lower magnetic field when the velocity of the disk is lower also. Note that error bars are the 2σ standard deduced from the linear fits to the temperature time series only; we didn't take into account the experimental errors due to measurements.

in the same way as the measurements shown in figure 3 except for one important difference: the dissipation derived from the torque is always about 3.5 larger than the one obtained from the temperature series. We could not find the reason for this systematic discrepancy, but because they are more direct, we regard the temperature measurements as being more accurate.

3.3 Induced magnetic field

Figure 4 shows the ratio of the induced magnetic field to the imposed field, as a function of the imposed field intensity, for several disk velocities. The two regimes are clearly identified. In the low field regime the efficiency of the induction decreases slowly as the magnetic field is increased up to $B_{imposed} \simeq 40$ mT, as a consequence of the decrease in vortex velocity. At values of $B_{imposed} > 40$ mT, the ratio decreases sharply with increasing imposed field as the vortex looses its two-dimensionality.

It was demonstrated in paper I that the ratio $B_{induced}/B_{imposed}$ is proportional to the effective magnetic Reynolds number, $Re_m = \mu_0 \sigma U_{eff} L$, where σ is the electrical conductivity, μ_0 the magnetic permeability, L a typical dimension (here the disk radius), and U_{eff} the typical effective velocity (here ω_{eff} L). We expect this relationship to remain valid over the parameter regime of this experiment as well. Accordingly, we use the relationship between $B_{induced}/B_{imposed}$ and Re_m to deduce the vortex velocity U_{eff}, using the calibration obtained in paper I.

4 Interpretation of the results

We wish to find a simple relationship between the amount of Joule heating, the fluid velocity in the vortex, and the intensity of the imposed field. Simple physical considerations suggest a relationship of the form

$$P_J = a \, \sigma \, U^2 \, B^2 \qquad (8)$$

where a is a constant which depends on geometry, σ is the electrical conductivity, U is the typical velocity, and B the typical magnetic field. This scaling can be obtained by combining the relationship between the current density $\vec{J}$ and the electro-motive-force:

$$J \simeq \sigma U B \qquad (9)$$

and the definition of Joule heating

$$P_J = \int_V \frac{J^2}{\sigma} dV \qquad (10)$$

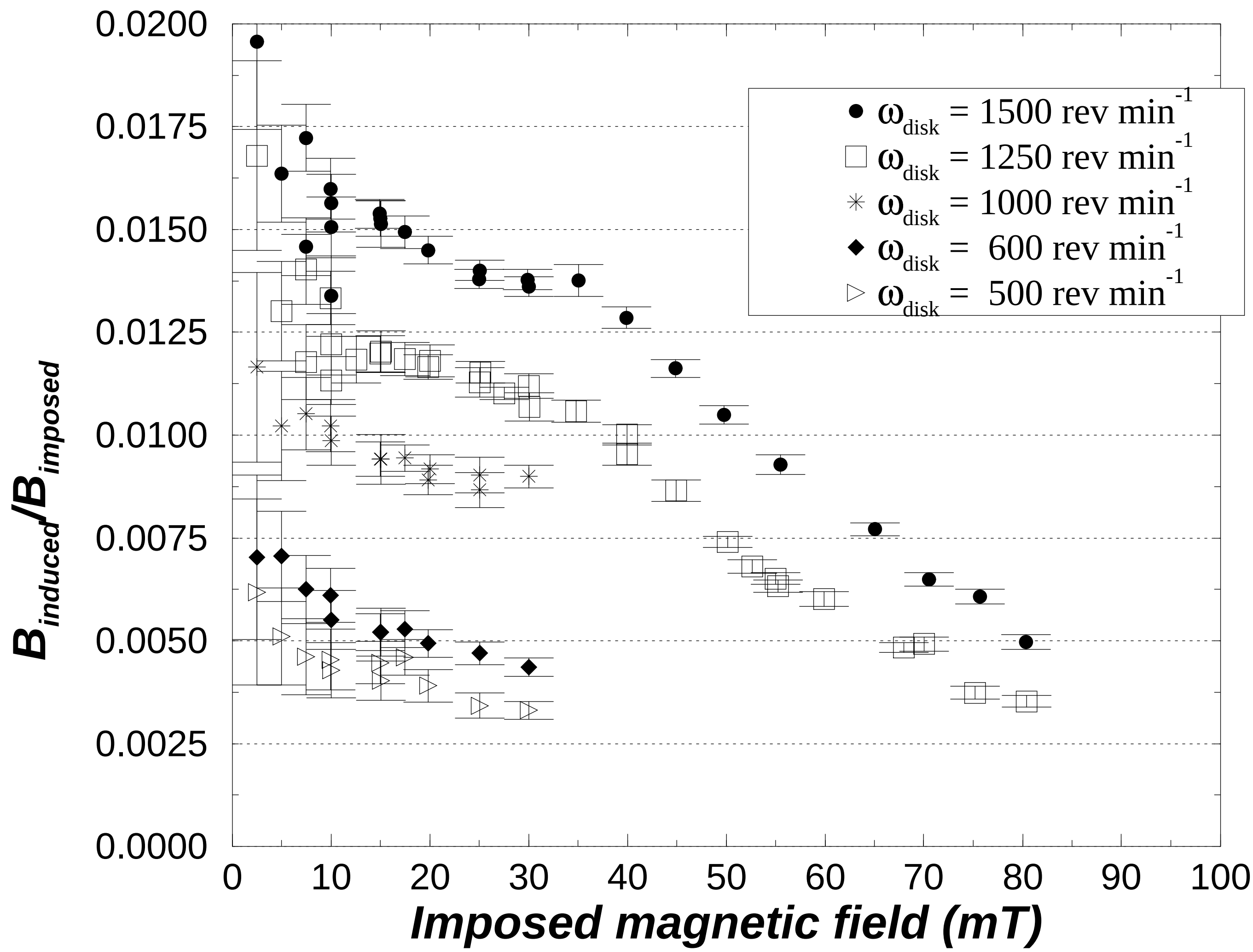

Figure 4: Induced magnetic field measured outside the cylinder during the experiments for different imposed magnetic fields from 2.5 mT to 78 mT, and for different velocities of the disk. The ratio $B_{induced}/B_{imposed}$ increases with the velocity of the disk. Note the break of slope for $\omega_{disk} = 1250$ and 1500 rev min^{-1} around 35-40 mT.

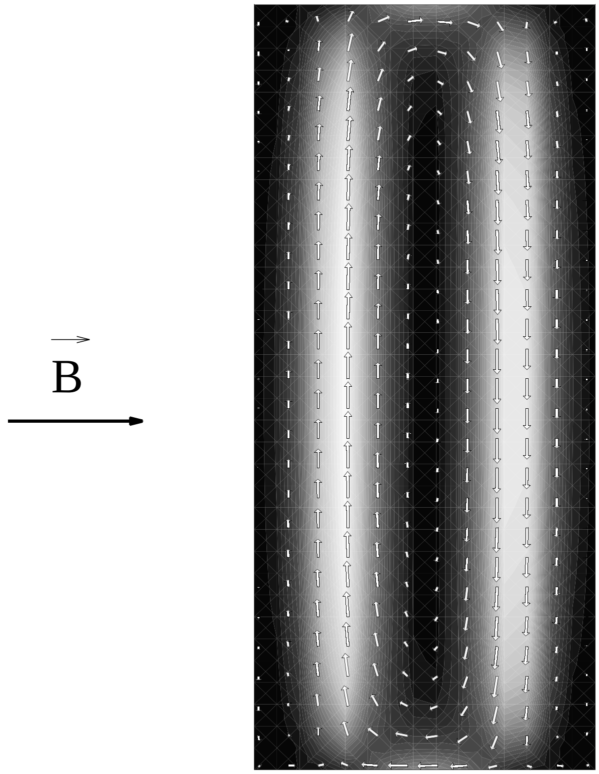

Figure 5: Numerical model of Foucault electrical currents in the cylinder. View of the vertical section of the cylinder that contains the imposed magnetic field $\vec{B}$. $\vec{J}$ is represented by arrows. Foucault electrical currents consist in a loop of current with upward currents on the left and downward current on the right. The colored areas are the distribution of the Joule heating in the cylinder: the brightest area (area of the biggest dissipation) is located along the loop of current where $J \simeq \sigma U B$ is maximum. This model depends upon two parameters: 1) R_{solid}, the radius of solid body rotation. R_{solid} determines the geometrical distribution of the electrical currents and consequently the total quantity of Joule heating in the tank. 2) ω_{solid}, the angular velocity of the solid body rotation.

The main assumption underlying (8) is that the magnetic Reynolds number of the flow is small. This is certainly the case in the experiment, but is problematic in the core. Because of this ambiguity, we examine two models of Joule heating. The first is appropriate for small magnetic Reynolds numbers, which yields the scaling given in (8). The second model, appropriate for the limit of very large magnetic Reynolds numbers, yields a somewhat different scaling law. We then compare the predictions of each law for dissipation in the Earth's core.

4.1 Joule heating at low magnetic Reynolds number

In paper I we introduced a simple 2D kinematic model of an MHD vortex, consisting of a core in a solid-body rotation plus a shear layer which adjusts the vorticity of the core to the vorticity of the container. There are only two parameters in this model, the radius R_{solid} of the vortex core, and its angular velocity ω_{solid}. In paper I, we also computed the distribution of the Foucault electrical currents generated by this simple vortex in an imposed horizontal magnetic field, taking into account the finite vertical extent of the cylinder. Figure 5 shows the distribution of Joule heating in the model, computed from the electrical currents. Joule heating mostly occurs along the loop of current close to r = R_{solid}, in the plane that contains $\vec{B}$, where the product $\vec{U} \times \vec{B}$ is maximum.

To obtain a scaling law between Joule heating and the other parameters, we will simplify the model further, by treating the vortex as if it were infinitely long and neglecting the $\vec{\nabla}\varphi$ term in the expression for the current density,

$$\vec{J} = \sigma(\vec{U} \times \vec{B} - \vec{\nabla}\varphi) \tag{11}$$

The Joule heating then becomes

$$\begin{aligned} P_J &= \int_V \frac{J^2}{\sigma} dV = \int_V [\sigma(\vec{U} \times \vec{B}]^2 r dr\ d\theta\ dz & (12) \\ &= H \int\int_S \sigma U^2(r) B^2 \cos^2\theta\ r\ dr\ d\theta & (13) \\ & & (14) \end{aligned}$$

Introducing R_{solid}= f R_{vortex} (with $0 \leq f \leq 1$), and using the velocity distribution in Appendix A of paper I, we get:

$$P_J = \sigma\, U(f)^2\, B^2\, V_{olume}\ \underbrace{[\frac{f^2}{4} + \frac{f^2}{(1-f^2)^2}(-\frac{3}{4} - \frac{f^4}{4} + f^2 + ln(\frac{1}{f}))]}_{C(f)} \tag{15}$$

According to (15), Joule heating varies as $\sigma\ U^2 B^2$, where U is the actual velocity for a given imposed B. In our model, U and P_J also depends upon the radius of solid body rotation. If we take $R_{solid} = R_{disk} = \frac{1}{2} R_{vortex}$ (i.e. f=$\frac{1}{2}$), as suggested by the results of paper I, we obtain

$$P_J = 1.3\ 10^{-4} \sigma U^2(B) B^2 \tag{16}$$

for the scaling law at low magnetic Reynolds number.

4.2 Joule heating in the experiments

The measurements in section 3.1 show the variation of Joule heating as a function of the imposed magnetic field, for various disk velocities. We now compare that data with the scaling

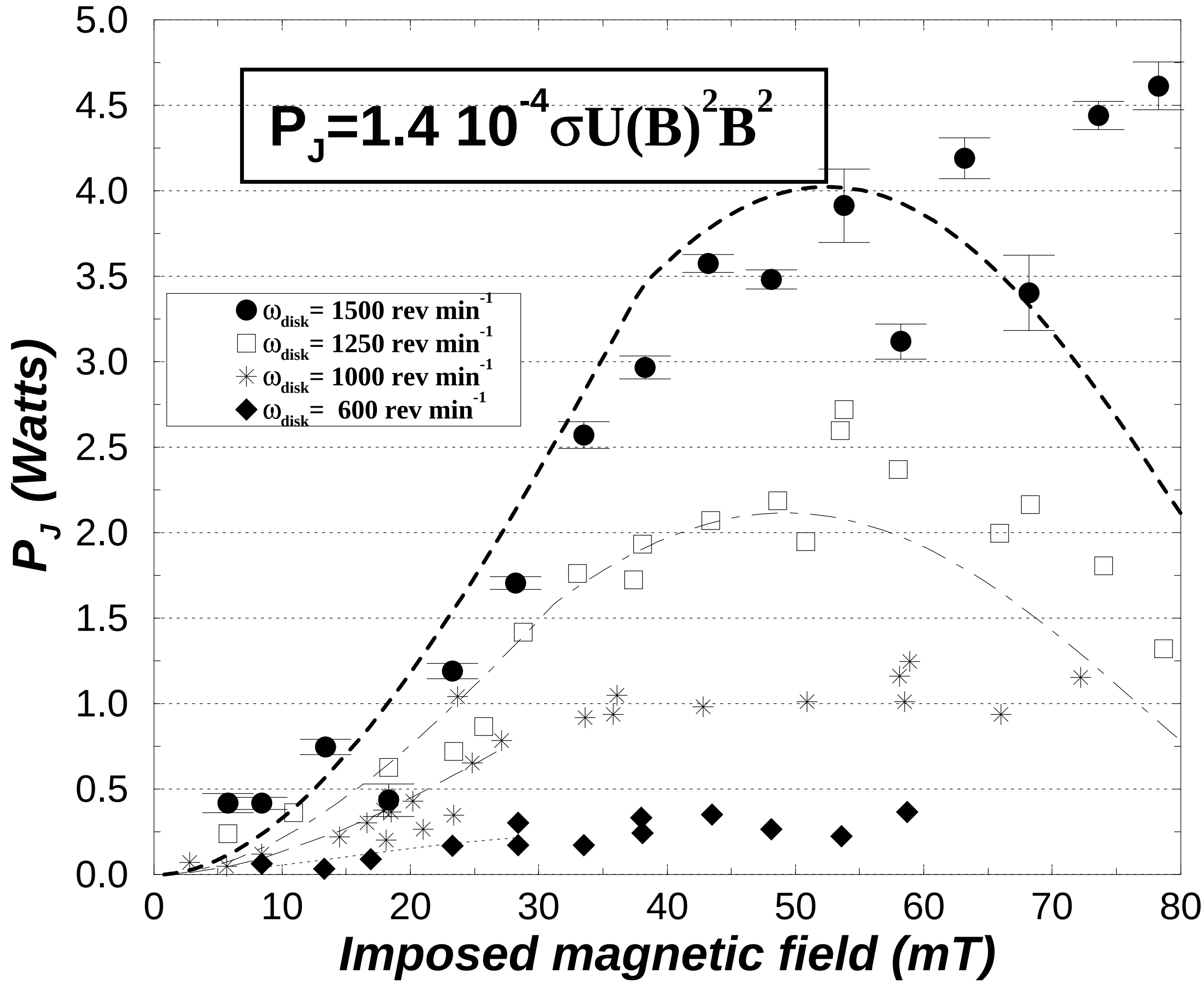

Figure 6: In dashed lined, fits of the experimental data set by a global scaling law for the Joule heating of type $P_J = a\sigma U(B)^2 B^2$. U(B) in this law is derived from the measurements of the induced magnetic field. The fit predicts reasonably the two regimes: a first quadratic increase of the Joule heating in U and B, and for the high applied magnetic fields, a slight decrease of the Joule heating due to the break of the vertical rigidity of the vortex.

law derived in the previous section. In order to make this comparison, we must relate the measured Joule heating to the actual fluid velocity in the vortex. To obtain the fluid velocity, we use the induced magnetic field shown in figure 4 in conjunction with the calibration between vortex velocity and induced magnetic field derived in paper I. In doing so, we implicitly neglect the effect of the small variation of the radius of solid body rotation found in paper I. In using this approach we are also implicitly assuming that the vortex remains two-dimensional. This assumption certainly breaks down for the highest values of the imposed magnetic field tested here. In addition, uncertainties on the position of the gaussmeter as compared to its location in paper I (about 2 mm) can result in errors in the induced field and velocity of about 20 %. With these approximations in mind, we proceed as follows. We first derive the velocity U(B) by fitting two segments of straight line to the curves $\frac{B_{induced}}{B_{imposed}} = f(B_{imposed})$. One segment is for the the first regime (up to to $B_{imposed}$=30 mT), the other one is for the regime where the induced field (and hence U) decreases more strongly. By comparing with the Joule heating measurements, we deduce the following experimental law:

$$P_J \simeq 1.4\ 10^{-4} \sigma U^2(B) B^2 \tag{17}$$

We can note the remarkable consistency between this experimental law and the law (16) derived from the 2-D model Figure (6) compares the actual experimental results with the predictions of this law (17). There is agreement in the first regime, where P_J is nearly quadratic in B. For higher values of B, our law predicts that Joule heating levels off and finally decreases with increasing B, in agreement with the set of measurements for $\omega_{disk} = 1250$ rev min^{-1}, but not so evident for other disk velocities. This drop in Joule heating, which is due to the strong reduction in fluid velocity by the magnetic field and strong departures from two-dimensionality in the vortex, is not as clear for the other values of the imposed disk velocity.

4.3 Joule heating at high magnetic Reynolds number

The maximum magnetic Reynolds number reached in our experiments is $R_m \simeq 0.3$ and the previous analysis is probably valid up to this value. For application to the Earth's core, however, it is necessary to consider how the relationship between Joule heating and imposed magnetic field changes at higher magnetic Reynolds numbers. Here we consider the limit of high magnetic Reynolds number, the regime where advection of magnetic field dominates over diffusion. In this regime the classical law in $\sigma U^2(B) B^2_{imposed}$ breaks down because the imposed magnetic field strength is no longer representative of the magnetic field within the vortex.

To derive the appropriate law for this regime, we follow Roberts (Gubbins and Roberts, 1987) who considers a uniform conductor filling the whole space, and everywhere at rest except for a cylindrical rotor of radius R_{rotor}, which spins about its vertical axis with the angular velocity ω. A uniform horizontal magnetic field $B_{imposed}$ is applied. Roberts (Gubbins and Roberts, 1987) demonstrates that when $Re_m \gg 1$, the magnetic field is expelled from the interior of the rotor, to form flux sheets near its surface. Roberts (Gubbins and Roberts, 1987) derived the analytical expression of the total magnetic field $\vec{B}_2$ just outside the rotor, as a function of the $B_{imposed}$, in the limit $Re_m \to \infty$. Note that in this limit the interior magnetic field $\vec{B}_1$ is zero.
For finite Re_m, the discontinuity of B will be smoothed on a length scale $\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$ called the skin.
The tangential components of the magnetic field are related to the surface electrical current $\vec{J}_s$, which develop on the surface of the rotor, by:

$$\vec{B}_1^T - \vec{B}_2^T = \mu_0(\vec{J}_s \times \vec{n}_{12}), \tag{18}$$

where $\vec{n}_{12}$ is the unit vector perpendicular to the rotor. Since the interior magnetic field $\vec{B}_1$ is zero, we deduce $\vec{J}_s$ from the expression of $\vec{B}_2^T$ given by Roberts (Gubbins and Roberts, 1987). Integrating J_s^2 along the rotor and over its height H_{rotor}, we get the expression for Joule heating :

$$P_{Jrotor} = \frac{4\pi B^2 R_{rotor} H_{rotor}}{\mu_0^2 \sigma \delta} \tag{19}$$

$$P_{Jrotor} = \frac{2\sqrt{2}\pi B^2 R_{rotor} H_{rotor} \sqrt{\omega}}{\mu_0^{\frac{3}{2}} \sigma^{\frac{1}{2}}} \tag{20}$$

Rewriting this in terms of the magnetic Reynolds number, we get:

$$P_{Jrotor} = 8.89 H_{rotor} B^2 \frac{\sqrt{Re_m}}{\mu_0^2 \sigma} \tag{21}$$

4.4 Ohmic dissipation as a function of the magnetic Reynolds number

The expression for Joule heating as a function of the actual fluid velocity in our experiments, rewritten in terms of the actual magnetic Reynolds number is

$$P_J = 1.96 \, H_{vortex} B^2 \frac{Re_m^2}{\mu_0^2 \sigma} \tag{22}$$

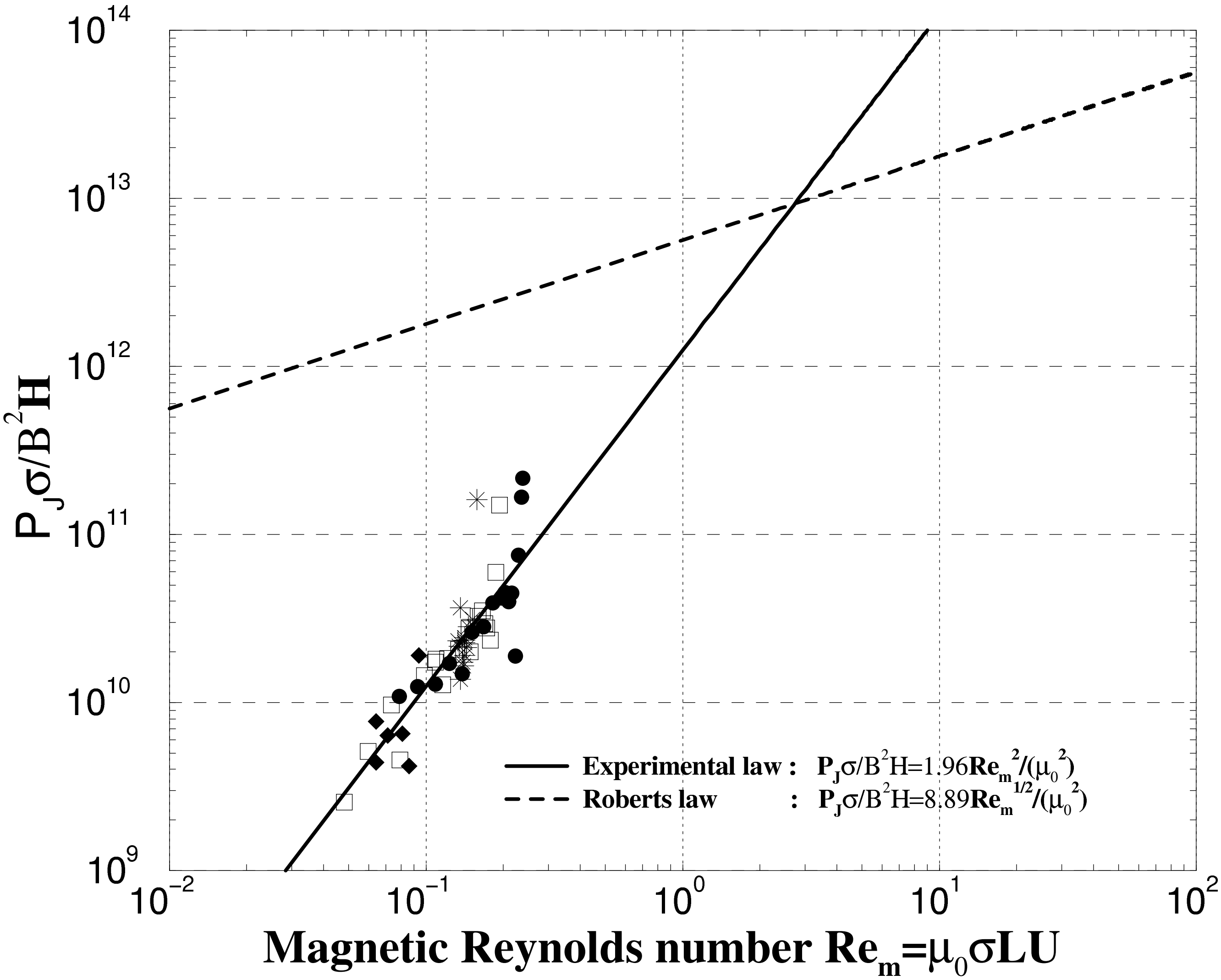

Figure 7: P_J is the total Joule heating (Watts) dissipated in a vortex; σ is the electrical conductivity of the fluid, H is the height of the vortex and B is the transverse magnetic field applied. The characteristic length scale used for the magnetic Reynolds number is the radius of the vortex and the characteristic velocity scale is the $U = \omega L$ where ω is the angular velocity of the solid body rotation of the vortex. The experimental law has been established for low magnetic Re_m: the experimental data points are represented with symbols. The convention used for these symbols is the same as in figure 3. The law of Roberts has been established for $Re_m \gg 1$. Consequently our global law for all the magnetic Reynolds number is probably not satisfactory around the point where the two law cross, around $Re_m = 2.7$. Elsewhere we can apply this law for estimations of the physical parameters of the Earth's core, supposing that thermal convection forms vortices in the Earth's core.

and is valid for low magnetic Reynolds number. For high magnetic Reynolds number we obtained the following expression

$$P_J = 8.89\ H_{vortex} B^2 \frac{\sqrt{Re_m}}{\mu_0^2 \sigma} \tag{23}$$

Figure 7 represents $P_J \sigma / H B^2$ as a function of Re_m. The laws corresponding to the two different Re_m regimes are shown, together with our experimental data points.

5 Application to Joule Heating in the Earth's Core

Using the formulas for dynamo efficiency from the introduction (3),

$$P_{Jcore} = \eta P_{CMB} \tag{24}$$

where P_J is Joule heat production in the Earth's core, P_{CMB} is the total heat flowing out of the core, and η is an efficiency factor. There is a considerable uncertainty on both η and P_{CMB}, but reasonable upper bounds seem to be (Lister et al., 1995):

$$\eta \leq 0.10 \tag{25}$$

$$P_{CMB} \leq 5\,TW \tag{26}$$

From which we get:

$$P_{Jcore} \leq 0.5 TW \tag{27}$$

This upper bound has often be used in the framework of kinematic dynamos to check the viability of the dynamo, and to put an upper bound on the intensity of the toroidal magnetic field in the core. Indeed, only the radial component of the poloidal part of the Earth's magnetic field can be obtained from measurements at the surface. Its maximum intensity at the surface of the core is about 0.5 mT. However, the constraint on the toroidal field from these analyses appears very weak: the kinematic dynamo of Pekeris et al. (1973) dissipates only 10^{-3} TW, with a toroidal field of 5 mT, while that of Kumar et al. (1975) dissipates 1.2 TW for a toroidal field of 200 mT. These dynamo models consider dissipation only at the largest (imposed) spatial scales of the velocity field. Our experiments suggest that this is not where most of the dissipation occurs. Instead, we find that dissipation by relatively small-diameter vertical vortices in the large-scale toroidal magnetic field could be predominant. The fact that the power spectrum of the poloidal field at the Core-Mantle-Boundary (Langel and Estes, 1982; Hulot and Le Mouël,

1994) is almost flat in the harmonic degree l (beyond the dominant dipole), and hence that the spectrum of B^2 rises sharply with l (at least up to l=13), also points towards dominant dissipation at short scales.

We apply our scaling law for Joule heating of vertical vortices in a uniform horizontal magnetic field to estimate the upper bound on the toroidal magnetic field using two models for core flow, Busse's convection-driven dynamo and the "observed" velocity field at the surface of the core. In the dynamo model of Busse (1970), thermal convection drives a circulation that takes the form of columnar vortices aligned with the axis of rotation (Taylor columns). The results of his analytical study indicate that the wavelength λ_c of this columnar instability in the equatorial plane is:

$$\lambda_c = (\frac{Pr\sqrt{(5/2)}}{2E(1+Pr)})^{\frac{1}{3}} \tag{28}$$

where Pr is the Prandtl number, and E the Ekman number. Taking for the Earth Pr=1 and E=10^{-15}, the number of columns is around 74000. Assuming that these 6200 km high narrow columns are arrayed within a uniform toroidal field B_T and that the magnetic Reynolds number of the columns exceeds 1, we derive from equation (17) and the upper bound (27) that B_T cannot exceed 0.6 mT. It is probably not possible to sustain the geodynamo with such a low toroidal field, which indicates that the wavelength of convection deduced from Busse's analysis is too small for the core.

The second situation is perhaps more relevant, because it is based upon the "observed" velocity field. Maps of fluid velocity at the top of the core have been obtained from the analysis of the secular variation of the magnetic field (Gire and Le Mouël, 1990; Bloxham et al., 1989; Jault, 1990). We assume that the velocity field of Hulot, Le Mouël and Jault (1990), for example, is the surface expression of geostrophic vortices extending through the core. To estimate the amount of Joule heating produced by such geostrophic vortices, we assume their flow consists of four large-diameter vortices tangent to the inner core, with height=5140 km, radius=1135 km, and velocity = 5 10^{-4} m/s, plus four smaller equatorial vortices each with height = 4000 km, radius = 800 km, and velocity = 3 10^{-4} m/s. The Joule heating of this array in a uniform toroidal field B_T can be calculated with the aid of equation (23). Using the upper bound (27), we find that B_T cannot exceed 9 mT for such a flow. Considering that we have computed the highest possible dissipation, and that smaller-scale features have not been included, the toroidal magnetic field in the Earth should be even smaller. On this basis, dynamo mechanisms that require a B_T/B_P ratio of more than about 20 are probably too dissipative for the Earth's core.

6 Conclusion

We have measured Joule dissipation in a vortex of liquid gallium permeated by a uniform transverse magnetic field, up to a magnetic Reynolds number of 0.3. The data indicate two regimes, which reflect the change in flow structure with increasing magnetic field intensity. For low magnetic field intensity the main effect of the Lorentz force is to reduce the fluid velocity while increasing the dimensions of the vortex; in this regime the Joule heating varies as the square of the imposed field. For very intense fields the Lorentz forces destroy the two-dimensionality of the vortex and the Joule heating becomes nearly independent of the applied field intensity. Using the induced magnetic field to calibrate the actual velocity in the vortex, we find a simple scaling law for Joule heating P_J, in the first regime with the form

$$P_J \simeq 2H_{vortex}B^2\frac{Re_m^2}{\mu_0^2\sigma} \tag{29}$$

where Re_m is the magnetic Reynolds number based on the fluid velocity. We compared this result with the Joule heating predicted in the asymptotic limit of high Re_m using the flux-expulsion model of Roberts (Gubbins and Roberts, 1987)

$$P_J \simeq 9H_{vortex}B^2\frac{\sqrt{Re_m}}{\mu_0^2\sigma} \tag{30}$$

We applied these two formulas to estimate Ohmic dissipation of geostrophic vortices in the large-scale magnetic field inside the Earth's core. We find that large Joule heating occurs by this mechanism. This places some limitation on the intensity of magnetic fields allowed within the core, since the total Joule heat production within the core is limited thermodynamically. For example, using an upper bound of 0.5 TW for Joule heating in the core and the model of Hulot et al. (1990) for core vortices, we find that the toroidal magnetic field in the core cannot be larger than 9 mT (about 20 times the observed poloidal field). Although this upper bound is ample to explain the geodynamo, it is less than some dynamo models predict. This indicates that Joule heat production can be used to constrain models of the geodynamo.

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