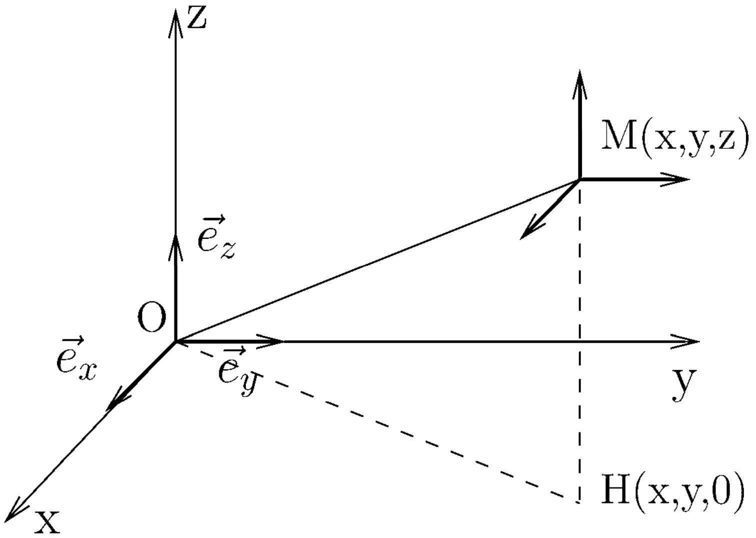
Master 1 des Sciences de la Terre, de l'Univers et de l'Environnement, Université Joseph-Fourier

U.E. TUE 408, Champs et Fluides Géophysiques, 2006/2007

TD Formulaire Mathématique

Opérateurs en coordonnées cartésiennes (x, y, z)



$$\overrightarrow{OM} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z$$

$$dV = dx\,dy\,dz$$

$$\overrightarrow{\text{grad}}\,P = \vec{\nabla} P = \begin{bmatrix} \dfrac{\partial P}{\partial x} \\ \dfrac{\partial P}{\partial y} \\ \dfrac{\partial P}{\partial z} \end{bmatrix}$$

$$\text{div}(\vec{u}) = \vec{\nabla} \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

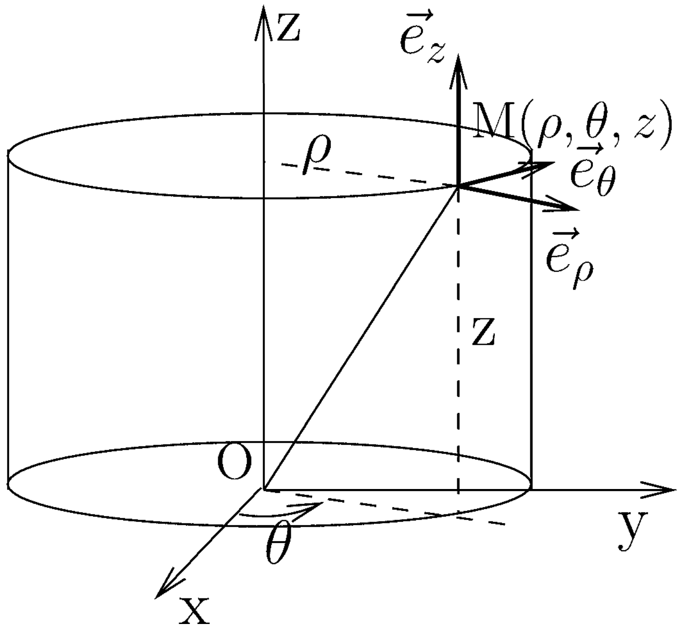
$$\overrightarrow{\text{grad}}\,\vec{u} = \vec{\nabla}\vec{u} = \begin{pmatrix} \dfrac{\partial u_x}{\partial x}, & \dfrac{\partial u_x}{\partial y}, & \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_y}{\partial x}, & \dfrac{\partial u_y}{\partial y}, & \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_z}{\partial x}, & \dfrac{\partial u_z}{\partial y}, & \dfrac{\partial u_z}{\partial z} \end{pmatrix}$$

$$\overrightarrow{\text{rot}}\,\vec{u} = \vec{\nabla} \times \vec{u} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_x}{\partial z} - \dfrac{\partial u_z}{\partial x} \\ \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} \end{bmatrix}$$

$$\Delta T = \vec{\nabla}^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\vec{\Delta}\,\vec{u} = \vec{\nabla}^2 \cdot \vec{u} = \begin{bmatrix} \dfrac{\partial^2 u_x}{\partial x^2} + \dfrac{\partial^2 u_x}{\partial y^2} + \dfrac{\partial^2 u_x}{\partial z^2} \\ \dfrac{\partial^2 u_y}{\partial x^2} + \dfrac{\partial^2 u_y}{\partial y^2} + \dfrac{\partial^2 u_y}{\partial z^2} \\ \dfrac{\partial^2 u_z}{\partial x^2} + \dfrac{\partial^2 u_z}{\partial y^2} + \dfrac{\partial^2 u_z}{\partial z^2} \end{bmatrix}$$

Opérateurs en coordonnées cylindriques (ρ, θ, z)



$$\overrightarrow{OM} = \rho \vec{e}_\rho + z \vec{e}_z$$

$$dV = \rho dr d\theta dz$$

$$\overrightarrow{\mathrm{grad}}\, P = \vec{\nabla} P = \begin{bmatrix} \dfrac{\partial P}{\partial \rho} \\[2ex] \dfrac{1}{\rho}\dfrac{\partial P}{\partial \theta} \\[2ex] \dfrac{\partial P}{\partial z} \end{bmatrix}$$

$$\mathrm{div}\, \vec{u} = \vec{\nabla} \cdot \vec{u} = \frac{1}{\rho}\frac{\partial(\rho u_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

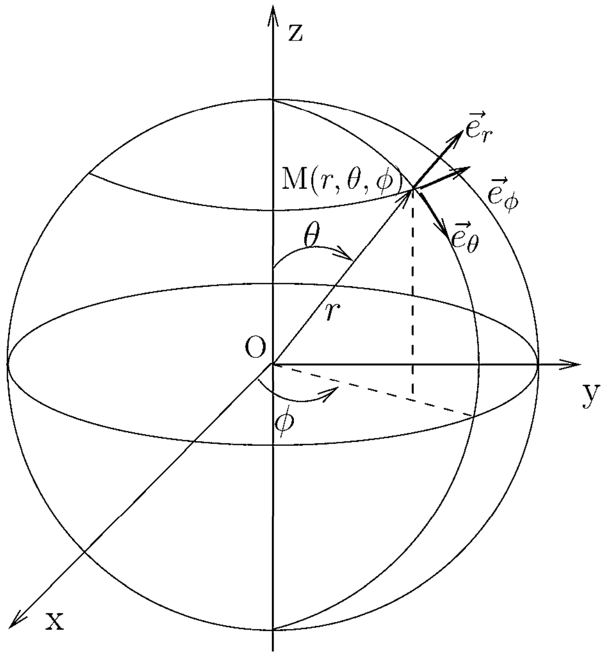
$$\overrightarrow{\mathrm{grad}}\, \vec{u} = \vec{\nabla} \vec{u} = \begin{pmatrix} \dfrac{\partial u_\rho}{\partial \rho}, & \dfrac{1}{\rho}\dfrac{\partial u_\rho}{\partial \theta} - \dfrac{u_\theta}{\rho}, & \dfrac{\partial u_r}{\partial z} \\[2ex] \dfrac{\partial u_\theta}{\partial \rho}, & \dfrac{1}{\rho}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_\rho}{\rho}, & \dfrac{\partial u_\theta}{\partial z} \\[2ex] \dfrac{\partial u_z}{\partial \rho}, & \dfrac{1}{\rho}\dfrac{\partial u_z}{\partial \theta}, & \dfrac{\partial u_z}{\partial z} \end{pmatrix}$$

$$\overrightarrow{\mathrm{rot}}\, \vec{u} = \vec{\nabla} \times \vec{u} = \begin{bmatrix} \dfrac{1}{\rho}\dfrac{\partial u_z}{\partial \theta} - \dfrac{\partial u_\theta}{\partial z} \\[2ex] \dfrac{\partial u_\rho}{\partial z} - \dfrac{\partial u_z}{\partial \rho} \\[2ex] \dfrac{1}{\rho}\dfrac{\partial(\rho u_\theta)}{\partial \rho} - \dfrac{1}{\rho}\dfrac{\partial u_\rho}{\partial \theta} \end{bmatrix}$$

$$\Delta T = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial T}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\rho}\frac{\partial T}{\partial \rho} + \frac{\partial^2 T}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\vec{\Delta} \vec{u} = \begin{bmatrix} \dfrac{\partial^2 u_\rho}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u_\rho}{\partial \rho} - \dfrac{u_\rho}{\rho^2} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u_\rho}{\partial \theta^2} + \dfrac{\partial^2 u_\rho}{\partial z^2} - \dfrac{2}{\rho^2}\dfrac{\partial u_\theta}{\partial \theta} \\[2ex] \dfrac{\partial^2 u_\theta}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u_\theta}{\partial \rho} - \dfrac{u_\theta}{\rho^2} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u_\theta}{\partial \theta^2} + \dfrac{\partial^2 u_\theta}{\partial z^2} + \dfrac{2}{\rho^2}\dfrac{\partial u_\rho}{\partial \theta} \\[2ex] \dfrac{\partial^2 u_z}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u_z}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u_z}{\partial \theta^2} + \dfrac{\partial^2 u_z}{\partial z^2} \end{bmatrix} = \begin{bmatrix} \Delta u_\rho - \dfrac{u_\rho}{\rho^2} - \dfrac{2}{\rho^2}\dfrac{\partial u_\theta}{\partial \theta} \\[2ex] \Delta u_\theta - \dfrac{u_\theta}{\rho^2} + \dfrac{2}{\rho^2}\dfrac{\partial u_\rho}{\partial \theta} \\[2ex] \Delta u_z \end{bmatrix}$$

Opérateurs en coordonnées sphériques (r, θ, ϕ)



$$\overrightarrow{OM} = r\,\vec{e}_r$$

$$dV = r^2 \sin\theta dr d\theta d\phi$$

$$\overrightarrow{\text{grad}}\, P = \vec{\nabla} P = \begin{bmatrix} \dfrac{\partial P}{\partial r} \\[2ex] \dfrac{1}{r}\dfrac{\partial P}{\partial \theta} \\[2ex] \dfrac{1}{r\sin\theta}\dfrac{\partial P}{\partial \phi} \end{bmatrix}$$

$$\text{div}(\vec{u}) = \vec{\nabla}\cdot\vec{u} = \frac{1}{r^2}\frac{\partial\left(r^2 u_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta u_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

$$\vec{\nabla}\vec{u} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r}, & \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r}, & \dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial\phi}\dfrac{u_\phi}{r} \\[2ex] \dfrac{\partial u_\theta}{\partial r}, & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r}, & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial\phi} - \dfrac{u_\phi \text{cotan}\theta}{r} \\[2ex] \dfrac{\partial u_\phi}{\partial r}, & \dfrac{1}{r}\dfrac{\partial u_\phi}{\partial\theta}, & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial\phi} + \dfrac{u_r}{r} + \dfrac{u_\theta \text{cotan}\theta}{r} \end{pmatrix}$$

$$\vec{\nabla}\times\vec{u} = \begin{bmatrix} \dfrac{1}{r\sin\theta}\left(\dfrac{\partial(\sin\theta u_\phi)}{\partial\theta} - \dfrac{\partial u_\theta}{\partial\phi}\right) \\[2ex] \dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial\phi} - \dfrac{1}{r}\dfrac{\partial(r u_\phi)}{\partial r} \\[2ex] \dfrac{1}{r}\dfrac{\partial(r u_\theta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta} \end{bmatrix}$$

$$\Delta T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

$$\vec{\Delta}\vec{u} = \begin{bmatrix} \Delta u_r - \dfrac{2}{r^2}u_r - \dfrac{2}{r^2\sin\theta}\dfrac{\partial(\sin\theta u_\theta)}{\partial\theta} - \dfrac{2}{r^2\sin\theta}\dfrac{\partial u_\phi}{\partial\phi} \\[2ex] \Delta u_\theta + \dfrac{2}{r^2}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r^2\sin^2\theta} - \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial u_\phi}{\partial\phi} \\[2ex] \Delta u_\phi + \dfrac{2}{r^2\sin\theta}\dfrac{\partial u_r}{\partial\phi} + \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial u_\theta}{\partial\phi} - \dfrac{u_\phi}{r^2\sin^2\theta} \end{bmatrix}$$

Si le champ de vecteur $\vec{u}$ est axisymétrique : $u_r(r,\theta)$, $u_\theta(r,\theta)$ et $u_\phi = 0$

$$\vec{\Delta}\vec{u} = \begin{bmatrix} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u_r}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u_r}{\partial\theta}\right) - \dfrac{2u_r}{r^2} - \dfrac{2}{r^2\sin\theta}\dfrac{\partial(\sin\theta u_\theta)}{\partial\theta} \\[2ex] \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u_\theta}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u_\theta}{\partial\theta}\right) + \dfrac{2}{r^2}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r^2\sin^2\theta} \\[2ex] 0 \end{bmatrix}$$

Relations entre vecteurs

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0$$

Relations entre opérateurs vectoriels

$$\vec{\nabla}\times(\vec{\nabla}T) = 0$$
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) = 0$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{u}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \vec{\Delta}\vec{u}$$
$$\vec{\nabla}\cdot(\vec{\Delta}\vec{u}) = \Delta(\vec{\nabla}\cdot\vec{u})$$
$$\vec{\nabla}(A\cdot B) = A\vec{\nabla}B + B\vec{\nabla}A$$
$$\Delta(AB) = A\Delta B + 2(\vec{\nabla}A)\cdot(\vec{\nabla}B) + B\Delta A$$
$$\vec{\nabla}\cdot(A\vec{u}) = A\vec{\nabla}\cdot\vec{u} + \vec{u}\cdot\vec{\nabla}A$$
$$\vec{\nabla}\times(A\vec{u}) = A\vec{\nabla}\times\vec{u} + (\vec{\nabla}A)\times\vec{u}$$
$$\vec{\nabla}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{\nabla}\times\vec{u}) - \vec{u}\cdot(\vec{\nabla}\times\vec{v})$$

Formes intégrales

Théorème de Gauss : l'intégrale de la divergence d'un vecteur sur un volume est égale à l'intégrale du flux sortant sur la surface fermée de ce volume.

$$\iiint_V \operatorname{div}\vec{u}\,dV = \iiint_V \vec{\nabla}\cdot\vec{u}\,dV = \oiint_S \vec{u}\cdot\vec{dS}$$

Théorème de Stokes : l'intégrale du rotationnel d'un vecteur sur une surface quelconque est égale à la circulation sur le contour fermé de cette surface.

$$\iint_S \vec{\operatorname{rot}}\,\vec{u}\cdot\vec{dS} = \iint_S \vec{\nabla}\times\vec{u}\cdot\vec{dS} = \oint_C \vec{u}\cdot\vec{dl}$$

Décomposition d'un vecteur

Un champ vecteur $\vec{u}$ quelconque peut se décomposer en la somme du gradient d'un potentiel scalaire U et du rotationnel d'un potentiel vecteur $\vec{\Psi}$:

$$\vec{u} = -\vec{\nabla}U + \vec{\nabla}\times\vec{\Psi}$$

Si la divergence de $\vec{u}$ est nulle alors :

$$\vec{u} = \vec{\nabla}\times\vec{\Psi}$$

Si le rotationnel de $\vec{u}$ est nul alors :

$$\vec{u} = -\vec{\nabla}U$$

Equation de Navier-Stokes

$$\rho\left(\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\vec{\nabla})\vec{u}\right)=-\vec{\nabla}P+\mu\vec{\Delta}\vec{u}+\vec{\mathcal{F}}$$

En coordonnées cartésiennes (x,y,z)

$$\begin{aligned}
\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right) &= -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)+\mathcal{F}_x \text{ selon } (Ox),\\
\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right) &= -\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right)+\mathcal{F}_y \text{ selon } (Oy),\\
\rho\left(\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right)+\mathcal{F}_z \text{ selon } (Oz).
\end{aligned}$$

En coordonnées cylindriques (r,θ,z)

$$\begin{aligned}
\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}\right) &= -\frac{\partial P}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}\right.\\
&\left.-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)+\mathcal{F}_r \text{ selon } (Or),\\
\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}\right) &= -\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{u_\theta}{r^2}\right.\\
&\left.+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right)+\mathcal{F}_\theta \text{ selon } (O\theta),\\
\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}+u_z\frac{\partial u_z}{\partial z}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right)\\
&+\mathcal{F}_z \text{ selon } (Oz).
\end{aligned}$$

En coordonnées sphériques (r,θ,ϕ)

$$\begin{aligned}
&\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\theta^2+u_\phi^2}{r}\right)=\\
&-\frac{\partial P}{\partial r}+\mu\left(\Delta u_r-2\frac{u_r}{r^2}-2\frac{\text{cotan}\theta}{r^2}u_\theta-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)+\mathcal{F}_r \quad \text{selon } (Or),\\
&\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{u_r u_\theta}{r}-\frac{u_\phi^2\text{cotan}\theta}{r}\right)=\\
&-\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left(\Delta u_\theta-\frac{u_\theta}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right)+\mathcal{F}_\theta \quad \text{selon } (O\theta),\\
&\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r u_\phi}{r}+\frac{u_\phi u_\theta\text{cotan}\theta}{r}\right)=\\
&-\frac{1}{r\sin\theta}\frac{\partial P}{\partial\phi}+\mu\left(\Delta u_\phi-\frac{u_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\right)+\mathcal{F}_\phi \quad \text{selon } (O\phi).
\end{aligned}$$